CSE 2500, Fall 2008 — Kiayias.

Pre-Final Review Exercises with Solutions. In addition to the exercises you solved so far and your notes, for the final you should also study the following exercises:

1. The degree-sequence of a graph is the sequence of the degrees of the vertices of the graph, i.e., $\langle d_1, \ldots, d_n \rangle$ where each d_i appears as a degree of a certain vertex in the graph. By convention we write the degree sequence so that $d_1 \geq \ldots \geq d_n$. Using properties of the degree sequence prove that any graph with $n \geq 2$ vertices has at least two vertices with the same degree.

   Proof. Suppose that the graph has no vertices with the same degree. It follows that since there are a total of n vertices the degree sequence of the graph must be of the form $\langle n-1, n-2, \ldots, 2, 1, 0 \rangle$ (because there are n vertices and all of them are of different degree). This is the case as the largest degree that a vertex can have is $n-1$ and the smallest degree is 0. Nevertheless, this sequence cannot be the degree sequence of a graph: if there was a graph with such a degree sequence it would have a vertex with degree $n-1$ and a vertex with degree 0. This is a contradiction.

2. Construct a connected graph with n vertices so that all its vertices have degree 2. Then show that any other connected graph that has n vertices that are all of degree 2 is isomorphic to the graph you constructed.

   Proof. Constructing a graph $G = (V, E)$ with n vertices so that all its vertices have degree 2 is fairly easy: consider the set of vertices $V = \{1, \ldots, n\}$ and set of n edges $\{\{i, i+1\} \mid i = 1, \ldots, n-1\} \cup \{1, n\}$. Verify that the degree of each vertext is 2.

   Now consider some other graph $G' = (V', E')$ so that all its vertices have degree two. Using the equality $\sum_{v \in V'} \text{degree}(v) = 2|E'|$ we derive the fact that $|E'| = n$, i.e., the graph G' has n edges. Suppose now that $V' = \{v_1, \ldots, v_n\}$. We will build a cycle of length n in G' which will immediately define the isomorphism to G.

   Let v_1 be the initial vertex; form a path K to be the vertext v_1. We will extend the path K vertex by vertex. Suppose that K contains any number k of vertices $n > k \geq 1$ and the last one added is called the "top." Pick a vertext $v' \in V' - K$ such that $\{v', v\} \in E'$ and $v \in K$. Such a vertex must exist otherwise the graph is not connected. Next we show that v is the top vertex in K. Indeed, if v is not the top it will hold that K has at least two elements and either there will be a vertex with degree 3 inside K or the vertex v has degree 1, both statements impossible given the theorem's assumption.

   It follows that K will grow to a path of length n. Then it follows that it has $n-1$ edges and the addition of the last remaining edge will make it a cycle (again we use the fact that all vertices have degree 2). It follows that G' is a cycle and thus isomorphic to G.

3. Define what is a bipartite graph. Then prove by induction on the number of vertices that all bipartite graphs of n vertices have at most $\frac{n^2}{4}$ edges.

   Proof. A bipartite graph is a graph $G = (V, E)$ that we can partition the set of vertices V into two sets L, R so that all edges of E are of the form $\{l, r\}$ where $l \in L$ and $r \in R$.

   We prove the statement by induction. Let us consider a base case for $n = 2$, i.e., two vertices. In this case the graph can have at most 1 edge. It holds that $\frac{2^2}{4} = 1$ thus the statement of the theorem holds.

Suppose now that for all bipartite graphs of n vertices it holds that the number of vertices is less than $\frac{n^2}{4}$. This is the induction hypothesis.

Now we proceed to the induction step as follows: suppose that G is a bipartite graph of $n+1$ vertices. The fact that G is bipartite implies that we can partition the set of vertices V into two sets L, R. If $|L| = |R|$ it follows that $n+1 = 2t$ and $|L| = |R| = t$. In this case it is immediate to verify that the number of edges can be at most t^2 i.e., $\frac{(n+1)^2}{4}$ (without employing the induction hypothesis).

Assume now without loss of generality that $|L| > |R|$ which implies $|R| < \frac{n+1}{2}$ which is in turn equivalent to $|R| \geq \frac{n}{2}$. Now Pick any vertex $v \in L$ and remove it from G together with all its incident edges. The resulting graph $G - \{v\}$ is also bipartite and morever it has n vertices, i.e., it satisfies that the number of edges is at most $\frac{n^2}{4}$. Now let us introduce v again into $G - \{v\}$ together will all its incident edges to obtain G. Since v cannot connect to any of the vertices in L it can only contribute up to $|R|$ edges to G. Since $|R| \leq \frac{n}{2}$ we have that the number of edges of G is at most

$$\frac{n^2}{4} + \frac{n}{2} < \frac{(n+1)^2}{4}$$

which completes the proof. □

4. Prove that any graph G that has $n \geq 2$ vertices and at least $\frac{(n-1)(n-2)}{2} + 1$ edges is necessarily connected.

   Proof. We will prove this fact by induction on n, the number of vertices. First we will consider the base case $n = 2$. Clearly any graph with two vertices and 1 edge is connected.

   The induction hypothesis will state: any graph G with a fixed number n of vertices such that $n \geq 2$ that additionally satisfies that the number of its edges is at least $\frac{(n-1)(n-2)}{2} + 1$ is a connected graph.

   Consider now a graph G with $n+1$ vertices where n is some fixed integer such that $n \geq 2$. Moreover we assume that the number of edges of G is at least $\frac{n(n-1)}{2} + 1$. Let us pick any vertex v from G that has degree non-zero and remove it. We are guaranteed that G contains such a vertex since in the other case it must be that the number of edges is 0, which is impossible given that $\frac{n(n-1)}{2} + 1 \geq 2$ for any $n \geq 2$.

   It holds that the number of edges of $G - \{v\}$ should be at least

$$\frac{n(n-1)}{2} + 1 - (n-1) = \frac{(n-1)(n-2)}{2} + 1$$

   (this is because there is a vertex v with degree of at most $n-1$, otherwise we have a fully-connected graph. Thus the removal of v can only incur a loss of $n-1$ edges from G). Based on the above it is clear that $G - \{v\}$ is a graph with n vertices and at least $\frac{(n-1)(n-2)}{2} + 1$ edges, as a result, based on the induction hypothesis the graph $G - \{v\}$ is connected. Since $G - \{v\}$ is connected and v has degree non-zero it holds that the graph G is connected as well. □

5. Let $V = \{1, \ldots, n\}$ and consider the following graphs:

(a) $G_1 = (V, E_1)$ where $E_1 = \{\{a, b\} \mid a + b = \text{odd}\}$.
(b) $G_2 = (V, E_2)$ where $E_2 = \{\{a, b\} \mid a + b = \text{even}\}$.
(c) $G_3 = (V, E_3)$ where $E_3 = \{\{a, b\} \mid a \cdot b = \text{odd}\}$.
(d) $G_4 = (V, E_4)$ where $E_4 = \{\{a, b\} \mid a \cdot b = \text{even}\}$.
(e) $G_5 = (V, E_5)$ where $E_5 = \{\{a, b\} \mid a \text{ divides } b\}$.
(f) $G_6 = (V, E_6)$ where $E_6 = \{\{a, b\} \mid a + b = \text{square}\}$.

For each one of $G_1, \ldots, G_6$, do the following: make a drawing of the graph find the connected components of the graph and decide whether (1) the graph has an Euler cycle, (2) the graph has a Hamilton cycle.

Proof. The above questions can be seen using the drawings. The below are explaining what you'll see:

(a) $G_1 = (V, E_1)$ where $E_1 = \{\{a, b\} \mid a + b = \text{odd}\}$.

Observe that if $a + b = \text{odd}$, then a and b have different parities, i.e. if a is even then b is odd. If there is a Hamilton cycle, let us traverse the vertices of that cycle by starting from a odd-numbered vertex(e.g. vertices 1, 3); this vertex is connected to an even-numbered vertex(e.g. vertices 2, 4) which is again connected to an odd-numbered vertex. Hence, there will be as many odd-numbered vertices as even-numbered vertices on that cycle, since there are n vertices on that cycle, it holds that n is even. Note also that this is also the sufficient requirement for G_1 to have Hamilton cycle.

Regarding the Euler cycle, we will try to check the degrees of vertices. Remember that if all vertices have even degree, then the graph has an Euler cycle. Any odd-numbered vertex is connected to any even-numbered vertex and vice versa. Hence degree of a odd-numbered vertex is equal to $\lfloor n/2 \rfloor$(this is number of even-numbered vertices), and similarly degree of a even-numbered vertex is equal to $\lceil n/2 \rceil$. Both $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ are even. This is only possible when n is divisible by 4.

(b) $G_2 = (V, E_2)$ where $E_2 = \{\{a, b\} \mid a + b = \text{even}\}$.

Observe that if $a + b = \text{even}$, then a and b have same parities, i.e. if a is even then b is even. Hence, odd-numbered vertices are connected to each other, and even-numbered vertices are connected to each other. Overall the graph G_2 has two components which fail G_2 to have Hamilton cycle or Euler Cycle.

(c) $G_3 = (V, E_3)$ where $E_3 = \{\{a, b\} \mid a \cdot b = \text{odd}\}$.

Observe that while odd-numbered vertices are connected to each other, the even-numbered vertices have degrees of zero. We have a graph that is consisting of many connected components, thus it fails to have Hamilton or Euler cycle.

(d) $G_4 = (V, E_4)$ where $E_4 = \{\{a, b\} \mid a \cdot b = \text{even}\}$.

Observe that if $a \cdot b = \text{even}$, then at least one of the vertices is even. If there is a Hamilton cycle, then any odd-numbered vertex is connected to even-numbered vertices from each side. Hence, even-numbered vertices are at least as many as the odd-numbered vertices, i.e. $\lfloor n/2 \rfloor \geq \lceil n/2 \rceil$. This is possible when n is even.

Regarding the Euler cycle: any even-numbered vertex is connected to everything else, i.e. degree of it is n, while the degree of an odd-numbered vertex is $\lfloor n/2 \rfloor$. Hence, n and $\lfloor n/2 \rfloor$ are even, which is possible when n is divisible by 4.

(e) $G_5 = (V, E_5)$ where $E_5 = \{\{a,b\} \mid a \text{ divides } b\}$.

Consider the largest prime number in the set $\{1, 2, \ldots, n\}$. You'll see that the vertex indexed with this prime number is connected to only vertex indexed with 1. Hence, the graph fails to have Hamilton and Euler Cycle.

(f) $G_6 = (V, E_6)$ where $E_6 = \{\{a,b\} \mid a + b = \text{square}\}$.

Regarding the Euler Cycle: Suppose that $a^2 \leq n < (a+1)^2$ holds for some a, i.e. $n = (a+1)^2 - x$ for a nonnegative integer $0 < x \leq 2a+1$. We have two cases for x: (i) if $x \geq 5$, then 1 is connected to $\{2^2 - 1, 3^2 - 1, \ldots, a^2 - 1\}$, while 4 is connected to only $\{3^2 - 4, \ldots, a^2 - 4\}$. Observe that the difference between the degrees of 1 and 4 is only one. Hence, if one is even then the other is not.

(ii) If $x \leq 4$, then 4 is connected to $\{3^2 - 4, \ldots, a^2 - 4, (a+1)^2 - 4\}$, while 9 is connected to $\{4^2 - 9, \ldots, a^2 - 9, (a+1)^2 - 9\}$. Observe that the difference between the degrees of 4 and 9 is one. Hence, they have different parities.

In any case, not all vertices have even degree; i.e. there is no Euler Cycle. (Be careful here that we assumed $n \geq 9$, for smaller values of n it is easy to check)

Regarding the Hamilton Cycle: we show that there are n for which there is no Hamilton cycle as follows. Recall that in a Hamilton cycle each vertex appears once and thus it holds that it can be represented as $\{a_1, a_2\}, \ldots, \{a_n, a_1\}$ where $(a_1 \ldots a_n)$ is a permutation of the vertices of the graph. Given the definition of the graph we have that $a_i + a_{i+1}$ is a square and thus it holds that $(a_1+a_2)+(a_2+a_3)+\ldots+(a_n+a_1)$ is a sum of n squares. Due to the fact that $(a_1 \ldots a_n)$ is a permutation we have that

$$(a_1 + a_2) + (a_2 + a_3) + \ldots + (a_n + a_1) = 2(1 + 2 + \ldots + n) = n(n+1)$$

and thus it should be the case that $n(n+1)$ is a sum of n squares. It is easy to find n for which this is impossible, e.g., $n = 5$.

6. A planar graph is a graph that can be drawn in the plane without its edges crossing (as an example consider K_4 which is planar vs. K_5 which is not planar). Structural induction is a general proof technique that enables us to show properties of objects that are defined inductively. An inductive definition defines a collection objects $\mathcal{G}$ by specifying an initial set of objects $\mathcal{G}_0$ and then a set of operations that act on the objects of $\mathcal{G}$ and produce new objects. The collection $\mathcal{G}$ is defined as the "closure" of these operations on $\mathcal{G}_0$. Structural induction is useful when we want to show that $\mathcal{G}$ has a certain property P: we first prove that the set of objects $\mathcal{G}_0$ satisfies P (usually one-by-one); this is the base case. Then we show that the operations that define $\mathcal{G}$ by acting inductively on $\mathcal{G}_0$ preserve the property P; this is the inductive step. Now consider the following example and exercise:

   Define the graph $G_0 = (V_0, E_0)$ with $V_0 = \{1,2,3\}$ and $E_0 = \{\{1,2\},\{2,3\},\{1,3\}\}$. Consider now the set of all graphs $\mathcal{G}$ that is defined as follows: $\mathcal{G}$ includes G_0 as well as any graph G' that is produced by applying to an element of $\mathcal{G}$ the following two operations:

   1. Given some $G = (V, E)$ pick any edge $\{i, j\} \in E$, select some $v \notin V$ and define $G' = (V \cup \{v\}, E \cup \{\{i, v\}, \{v, j\}\})$.
   2. Given some $G = (V, E)$ pick any edge $\{i, j\} \in E$, and define the graph $G' = (V, E - \{i, j\})$.

Prove by structural induction that all graphs of the family $\mathcal{G}$ are planar.

Proof. We first prove the base case, i.e., that G_0 is planar. Indeed it is very easy to see that G_0 is simply a "triangle" cycle graph and it is planar.

Suppose now that G is a planar graph. We will show that by applying the two operations that define the family $\mathcal{G}$ we can only get planar graphs. Suppose we have a planar drawing of G. If we apply operation 2 on G the resulting graph is clearly planar as well (as we simply erased one edge, this has no effect in the planarity of G).

Next we consider operation 1 as applied to the planar graph G. Since G is a planar graph it holds that every edge $\{i, j\}$ belongs to the boundary of two faces of the graph. In the resulting graph G' we will simply be adding a new vertex v and two new edges $\{i, v\}$ and $\{j, v\}$. It is easy to get a planar drawing for G': simply choose at random one of the two faces on which the edge $\{i, j\}$ is the boundary and place the new vertex v somewhere in this face. Then, draw the two edges connecting v with i and j. Since the face is a Jordan curve (non self-intersecting closed curve) it holds that we can connect any point of the interior of the face to two points in the boundary of the curve. It follows that the resulting drawing is planar.

7. Consider urn A that initially contains n red balls and an urn B that initially contains n blue balls. At any move we choose one ball at random from urn A, we discard it and then we transfer one ball from urn B to urn A (if any are left). Balls are indistinguishable except for color. We continue this experiment till there are no more balls in both urns. In how many steps will this process terminate? What is the probability that the last ball we remove from urn A is red?

   Proof. To gain some intuition you can imagine the whole process as the following two experiments executed consecutively:

   1) in the first one we have n balls in front of us. Initially they are all red. We grab a blue ball in hand, we pick one of the n balls us at random and we substitute it. This is repeated n times. Note that if we happen to pick a red ball we switch it with blue. If we happen to pick a blue ball we switch it with blue again (so nothing really happens).

   2) in the second experiment we have n balls in front of us [some of them blue some of them red] with an empty hand we pick a ball at random and we discard it.

   We want to find the probability that the last ball we pick is red.

   Observe now the following: the first experiment is like picking a random function f from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. In particular $f(i)$ would correspond to "the location of the ball that was switched with a blue one in the i-th move."

   The second experiment is like picking a random permutation π from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. In particular $\pi(i) =$ "the location of the ball I discarded in the i-th move."

   Based on the above, the event we seek to compute the probability of is "for all i: $f(i) \neq \pi(n)$" over all possible choices of f, π. The total number of possibilities is $n!n^n$. Then, fix an arbitrary permutation π: the total number of ways to select a function f so that the event happens is $(n-1)^n$ (we simply exclude $\pi(n)$ - which is fixed - from the range of f). It follows that the required probability is $n!(n-1)^n/(n!n^n) = (1-1/n)^n$. □

8. In a production line, a box containing n items is discarded if after sampling $k < n$ items at random from the box one of them is found faulty. Suppose that a box contains $t < n$ faulty items. What is the probability that the box is detected to be faulty?

   Solution. We have a total of n items out of which t are faulty. The number of ways we can select k items from the box so that the box is not detected as faulty is $\binom{n-t}{k}$ (provided that $n - t \geq k$; what happens if $n - t < k$?). It follows that the probability that the box is detected as faulty is $1 - \frac{\binom{n-t}{k}}{\binom{n}{k}}$.

9. Suppose you a deck of cards (52 cards) is randomly shuffled. Then the deck is cut exactly in two parts, left and right. What is the probability that the ace of spades is in the left pack and the king of hearts is in the right pack (event A)? What is the probability that the ace of spades and the king of hearts is in the left pack (event B)? Suppose now you flip the first card of the left pack open and it turns out it is the king of spades (event C). What is the probability of the event A now? What is the probability of the event B? What can you say about the dependency or independence of the events A, B and C?

   Solution. The probability of the event A is $26^2/52 \cdot 51 \approx 25.4\%$. The probability of the event B is $26 \cdot 25/52 \cdot 51 \approx 24.5\%$. Next we compute the conditional probability of the event A conditioned on the event C, i.e. $\mathbf{Pr}[A \mid C] = \frac{\mathbf{Pr}[A \cap C]}{\mathbf{Pr}[C]}$. It turns out equal to the probability of the event A. This means that the events A and C are independent, since $\mathbf{Pr}[A \cap C] = \mathbf{Pr}[A] \cdot \mathbf{Pr}[C]$ holds. Next we compute the conditional probability of B conditioned on C. It turns out it is about 23.5%. It follows that the events B and C are not independent (in particular after observing C we should loose confidence on B somewhat).

10. Consider the following casino game called CRAZY-100: In order to play you bet \$ 6. Three 6-sided dice are rolled. If all three outcomes are equal you win \$ 100 (on top of your bet). If just two outcomes are equal you get back your \$ 6. Otherwise you loose. Would you play the game (calculate the expected winnings) ?

    Proof. Let the random variable X be the money you receiver after the dices are rolled. X equals to \$100 if all three dices outcome same number(with probability 6/216, why?), and X equals to \$0 if two of them outcome same((with probability 90/216), why?), and X equals to -6 otherwise(remaining probability 20/36). The random variable is defined as follows:

$$X(s) = \begin{cases} 100, & \text{with probability } 1/36 \\ 0, & \text{with probability } 15/36 \\ -6, & \text{with probability } 20/36 \end{cases}$$

    Compute expectation from above.

11. Consider the complete graph with n vertices K_n. An edge-coloring of K_n is an assignments of colors (say blue or red) to its edges. Consider an edge-coloring of K_n; an edge-coloring of K_n is called "k-interesting" if, under the coloring, K_n has no single colored K_k subgraph (a single colored subgraph of K_n is a subgraph where all its edges are either red or blue). Find a sufficient condition on n, k so that the graph K_n has a k-interesting edge-coloring. Hint : define a probabilistic manner for coloring the graph

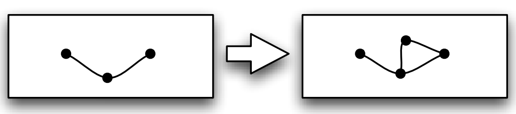
so that the probability of the event that a coloring of K_n which has a k-interesting edge-coloring is non-zero.

Proof. First we will define a probability space over all edge-colorings. The probability space will be simply painting each edge red or blue at random (with probability 1/2 each color). Then we will investigate the probability of producing a single-color subgraph K_k (either totally red or totally blue).

Now we fix some K_k subgraph of K_n (note: there are $\binom{n}{k}$ such subgraphs). The probability it is painted with a single color is $2/2^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$. Since there is a total of $\binom{n}{k}$ subgraphs like this, the probability that one of them is painted with a single color cannot be larger than $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$.

Since we want to produce an edge-coloring for which no single-colored K_k subgraph exists we conclude that a sufficient condition for such a coloring to exist is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$. This condition provides a lower bound on the probability of producing a k-interesting coloring by assigning colors to edges at random. Observe that k-interesting colorings cannot be produced for too small k's in general.

12. The game of "Sprouts" is played as follows: n dots are placed arbitrarily on a paper. The two players take turns performing a valid move that is defined as follows: An edge is drawn between two dots provided they have degree less than 3 (dots with degree three are "full"). Whenever such an edge is drawn by one of the players a new dot must also be drawn in the middle of the connecting edge. Crossing between edges is *not allowed*. The player that moves last wins. A valid move is depicted below:



Prove that a game with n initial dots lasts no more than $3n-1$ moves (for any strategy of the players). Then, find the number of all different Sprouts games for initial configuration of $n = 2$ points (you can draw the configuration graph). Finally show that player 2 has a winning strategy!

Solution. A player moves by drawing an edge between two existing points and introducing a new vertex on the new edge. The new edge will capture two degrees of freedom from the existing graph (since it connects to two verticies) and add only one degree of freedom — the new vertex has a degree two.

From the discussion above it is clear that any move reduces the degree of freedom in the graph by 1. Moreover, when the graph has degree of freedom less than two then the game ends. Observe finally that in the initial configuration there are $3n$ degrees of freedom (since there are n points without any connections). Therefore, the maximum number of moves is at most $3n - 1$.

For the second part we will produce the graph of game configurations. There are 46 different paths in this game, thus there are 46 different sprouts games. By labelling the graph vertices with P-positions and N-positions we conclude that the initial configuration (top vertex) is a P-position thus player 2 has a winning strategy.

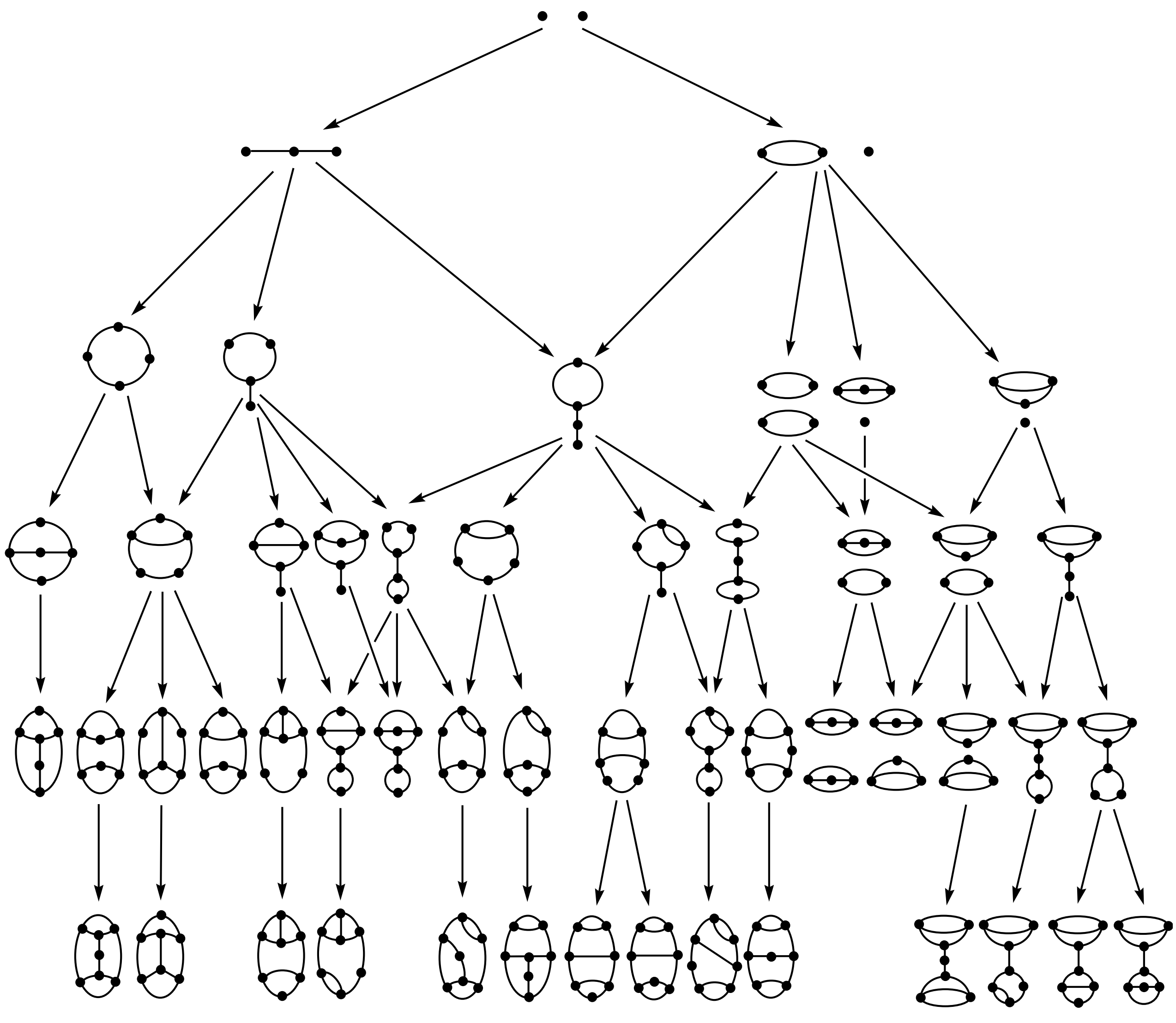


Figure 2: The complete game tree for two-spot Sprouts

Figure 1: The above figure is taken from — D. Applegate and G. Jacobson titled “Computer Analysis of Sprouts”—technical report, Carnegie Mellon University, CMU-CS-91-144.