CSE 2500, Fall 2008 — Kiayias.

Pre-Final Review Exercises with Solutions. In addition to the exercises you solved so far and your notes, for the final you should also study the following exercises.

> **You should also do exercise 12 as homework and deliver it in class on December 4, 2008.**

1. The degree-sequence of a graph is the sequence of the degrees of the vertices of the graph, i.e., $\langle d_1, \ldots, d_n \rangle$ where each d_i appears as a degree of a certain vertex in the graph. By convention we write the degree sequence so that $d_1 \geq \ldots \geq d_n$. Using properties of the degree sequence prove that any graph with $n \geq 2$ vertices has at least two vertices with the same degree.

2. Construct a connected graph with n vertices so that all its vertices have degree 2. Then show that any other connected graph that has n vertices that are all of degree 2 is isomorphic to the graph you constructed.

3. Define what is a bipartite graph. Then prove by induction on the number of vertices that all bipartite graphs of n vertices have at most $\frac{n^2}{4}$ edges.

4. Prove that any graph G that has $n \geq 2$ vertices and at least $\frac{(n-1)(n-2)}{2} + 1$ edges is necessarily connected.

5. Let $V = \{1, \ldots, n\}$ and consider the following graphs:

   (a) $G_1 = (V, E_1)$ where $E_1 = \{\{a, b\} \mid a + b = \text{odd}\}$.

   (b) $G_2 = (V, E_2)$ where $E_2 = \{\{a, b\} \mid a + b = \text{even}\}$.

   (c) $G_3 = (V, E_3)$ where $E_3 = \{\{a, b\} \mid a \cdot b = \text{odd}\}$.

   (d) $G_4 = (V, E_4)$ where $E_4 = \{\{a, b\} \mid a \cdot b = \text{even}\}$.

   (e) $G_5 = (V, E_5)$ where $E_5 = \{\{a, b\} \mid a \text{ divides } b\}$.

   (f) $G_6 = (V, E_6)$ where $E_6 = \{\{a, b\} \mid a + b = \text{square}\}$.

   For each one of $G_1, \ldots, G_6$, do the following: make a drawing of the graph find the connected components of the graph and decide whether (1) the graph has an Euler cycle, (2) the graph has a Hamilton cycle.

6. A planar graph is a graph that can be drawn in the plane without its edges crossing (as an example consider K_4 which is planar vs. K_5 which is not planar). Structural induction is a general proof technique that enables us to show properties of objects that are defined inductively. An inductive definition defines a collection objects $\mathcal{G}$ by specifying an initial set of objects $\mathcal{G}_0$ and then a set of operations that act on the objects of $\mathcal{G}$ and produce new objects. The collection $\mathcal{G}$ is defined as the "closure" of these operations on $\mathcal{G}_0$. Structural induction is useful when we want to show that $\mathcal{G}$ has a certain property P: we first prove that the set of objects $\mathcal{G}_0$ satisfies P (usually one-by-one); this is the base case. Then we show that the operations that define $\mathcal{G}$ by acting inductively on $\mathcal{G}_0$ preserve the property P; this is the inductive step. Now consider the following example and exercise:

Define the graph $G_0 = (V_0, E_0)$ with $V_0 = \{1, 2, 3\}$ and $E_0 = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Consider now the set of all graphs $\mathcal{G}$ that is defined as follows: $\mathcal{G}$ includes G_0 as well as any graph G' that is produced by applying to an element of $\mathcal{G}$ the following two operations:

1. Given some $G = (V, E)$ pick any edge $\{i, j\} \in E$, select some $v \notin V$ and define $G' = (V \cup \{v\}, E \cup \{\{i, v\}, \{v, j\}\})$.
2. Given some $G = (V, E)$ pick any edge $\{i, j\} \in E$, and define the graph $G' = (V, E - \{i, j\})$.

Prove by structural induction that all graphs of the family $\mathcal{G}$ are planar.

7. Consider urn A that initially contains n red balls and an urn B that initially contains n blue balls. At any move we choose one ball at random from urn A, we discard it and then we transfer one ball from urn B to urn A (if any are left). Balls are indistinguishable except for color. We continue this experiment till there are no more balls in both urns. In how many steps will this process terminate? What is the probability that the last ball we remove from urn A is red?

8. In a production line, a box containing n items is discarded if after sampling $k < n$ items at random from the box one of them is found faulty. Suppose that a box contains $t < n$ faulty items. What is the probability that the box is detected to be faulty?

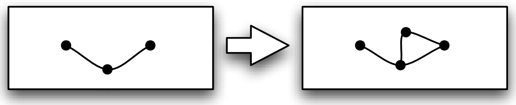
9. Suppose you a deck of cards (52 cards) is randomly shuffled. Then the deck is cut exactly in two parts, left and right. What is the probability that the ace of spades is in the left pack and the king of hearts is in the right pack (event A)? What is the probability that the ace of spades and the king of hearts is in the left pack (event B)? Suppose now you flip the first card of the left pack open and it turns out it is the king of spades (event C). What is the probability of the event A now? What is the probability of the event B? What can you say about the dependency or independence of the events A, B and C?

10. Consider the following casino game called CRAZY-100: In order to play you bet \$ 6. Three 6-sided dice are rolled. If all three outcomes are equal you win \$ 100 (on top of your bet). If just two outcomes are equal you get back your \$ 6. Otherwise you loose. Would you play the game (calculate the expected winnings) ?

11. Consider the complete graph with n vertices K_n. An edge-coloring of K_n is an assignments of colors (say blue or red) to its edges. Consider an edge-coloring of K_n; an edge-coloring of K_n is called "k-interesting" if, under the coloring, K_n has no single colored K_k subgraph (a single colored subgraph of K_n is a subgraph where all its edges are either red or blue). Find a sufficient condition on n, k so that the graph K_n has a k-interesting edge-coloring. Hint : define a probabilistic manner for coloring the graph so that the probability of the event that a coloring of K_n which has a k-interesting edge-coloring is non-zero.

12. The game of "Sprouts" is played as follows: n dots are placed arbitrarily on a paper. The two players take turns performing a valid move that is defined as follows: An edge is drawn between two dots provided they have degree less than 3 (dots with degree three are "full"). Whenever such an edge is drawn by one of the players a new dot must also

be drawn in the middle of the connecting edge. Crossing between edges is *not allowed*. The player that moves last wins. A valid move is depicted below:



Prove that a game with n initial dots lasts no more than $3n-1$ moves (for any strategy of the players). Then, find the number of all different Sprouts games for initial configuration of $n = 2$ points (you can draw the configuration graph). Finally show that player 2 has a winning strategy!