

CSE 2500 — Kiayias.

Pre-Midterm Review Exercises with Solutions (except 11-12 due monday).

In addition to the exercises you solved so far, we will also discuss the following exercises in class.

1. Let $F(0) = 0, F(1) = 1, F(2) = 1, F(3) = 2, \dots$ be the Fibonacci sequence. Prove that for all $n \geq 1$ it holds that $F(3n)$ is an even number.

Proof. Consider the case $n = 1$. It is easy to see that $F(3)$ is an even number. Assume (induction hypothesis) that $F(3n)$ is an even number. We will prove that $F(3(n + 1))$ is an even number. Indeed $F(3(n + 1)) = F(3n + 3) = F(3n + 1) + F(3n + 2) = F(3n + 1) + F(3n + 1) + F(3n) = 2F(3n + 1) + F(3n)$ which is an even number based on the induction hypothesis. \square

2. Recall that functions $X \rightarrow Y$ can be seen as sets of pairs, i.e., subsets of $X \times Y$. If $f, g \subseteq X \times Y$ are functions is it true that $f \cup g$ and $f \cap g$ are also functions?

Hint. No they will not be. In particular $f \cap g$ will not necessarily be defined for all elements of the domain X whereas $f \cup g$ will likely correspond two different values for the same x -value. \square

3. Suppose you begin with a file of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute $r \cdot s$. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

Proof. Consider the case when $n = 2$. It is easy to observe that there will be only one step resulting in two piles of one stone each. The sum of the products will be $1 = 2 \cdot (2 - 1)/2$. Assume (strong induction hypothesis) that if the number of stones is less than or equal to n , the sum of products computed at each step is $n(n - 1)/2$. We will compute, no matter how you split the piles, the sum for $n + 1$ stones is $(n + 1)n/2$. Consider the first split is done so that $r + s = n + 1$. The product at this step is rs , the sum of the products for the pile with r stones is $r(r - 1)/2$, and the sum of the products for the pile with s stones is $s(s - 1)/2$. Sum of these sums will yield the result.

4. (a) If $g \circ f$ is an onto function, does g have to be onto? Does f have to be onto? (b) If $g \circ f$ is an one-to-one function, does g have to be one-to-one? Does f have to be one-to-one?

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Note that $g(f(x))$ is an element of Z .

(a) Suppose $g \circ f$ is onto. this means that for all $z \in Z$ we have that there is a $x \in X$ such that $g(f(x)) = z$. This implies that there is also a $y \in Y (= f(x))$ so that $g(y) = z$ and thus g is also onto. On the other hand f may not be onto, i.e., there can be elements in Y for which there is no x such that $f(x) = y$. e.g. $f : 1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b$; and $X = \{1, 2, 3\}, Y = \{a, b, c\}$ (f is not onto). Now consider the function $g : \{a, b, c\} \rightarrow \{7, 8\}$ so that $g(a) = g(c) = 7$ and $g(b) = 8$. Clearly $g \circ f$ is onto since it is the function that maps $g(f(1)) = 7$ and $g(f(3)) = 8$.

(b) Suppose now $g \circ f$ is 1-1. This means that for all x_1, x_2 , with $x_1 \neq x_2$, we have that $g \circ f(x_1) \neq g \circ f(x_2)$. Suppose also that f is NOT 1-1. Then it holds that there are x_1, x_2 with $x_1 \neq x_2$, so that $g \circ f(x_1) = g \circ f(x_2)$ so $g \circ f$ is also NOT 1-1, a contradiction. Thus if $g \circ f$ is 1-1 this means that f is 1-1. Now what happens with g ? If g is not 1-1 this means that $g(y_1) = g(y_2)$ for some y_1 different from y_2 . Now if there exist x_1 and x_2 such that $f(x_1) = y_1$ and $f(x_2) = y_2$ we have a contradiction. but f may be not onto. So it is possible that g is not 1-1 as long as its “ambiguity” falls into places that f does not cover. \square

5. For a relation R on a set X we define the symbol R^n as follows: $R^1 = R$ and $R^{n+1} = R \circ R^n$ (see definition 6 of Chapter 8.1 in the text).

(a) Prove that if X is finite and R is a relation on it, then there exist $r, s \in \mathbb{N}$ $r < s$ such that $R^r = R^s$.

(b) Find a relation R on a finite set such that $R^n \neq R^{n+1}$ for every $n \in \mathbb{N}$.

Proof. (by contradiction) Suppose that for all $s > t$ it holds that R^s and R^t are different. Then, R, R^2, R^3, \dots are all distinct relations.

This is a contradiction. The number of relations over X are finite! (how many? count... a relation is a subset of $X \times X$ i.e., there are $2^{|X|^2}$ relations.

For (b) consider $\{a, b\}$ and the relation (a, b) and (b, a) .

Observe that R^2 contains (a, a) and (b, b) .

On the other hand, R^3 is equal to R etc... \square

6. Formulate the conditions for reflexivity of a relation, for symmetry of a relation, and for its transitivity using the adjacency matrix of the relation.

Proof If $a_{i,j}$ is the i -th row, j -th column element of the adjacency matrix A , the transitive condition suggests that $a_{i,k} = 1$ and $a_{k,j} = 1$ then $a_{i,j} = 1$.

Observe that the i -th row j -th column element of $A * A$ is equal to $a'_{i,j} = \sum_k a_{i,k} \cdot a_{k,j}$.

Now if $a_{i,k} \cdot a_{k,j} = 1$ for some k then $a'_{i,j} \geq 1$. If the relation is transitive it should be that for each $a'_{i,j}$ with the property $a'_{i,j} \geq 1$ it must be that $a_{i,j} = 1$ [OTHERWISE it is not transitive]. \square

7. Call an equivalence \sim on the set \mathbb{Z} (the integers) a *congruence* if the following condition holds for all $a, x, y \in \mathbb{Z}$: if $x \sim y$ then also $a + x \sim a + y$.

(a) Let q be a nonzero integer. Define a relation \equiv_q on \mathbb{Z} by letting $x \equiv_q y$ if and only if q divides $x - y$. Check that \equiv_q is a congruence according to the above definition.

(b) What are the equivalence classes of the congruence \equiv_q ?

hint. (a) Note that $(a + x) - (a + y) = x - y$, and if q divides right-hand side, then it also divides left-hand side. (b) Consider the sets $\{0, q, 2q, \dots\}$, $\{1, q + 1, 2q + 1, \dots\}$ and so on. \square

8. Cars are compared according to two properties: gas consumption per 100 miles m and acceleration a . A car is represented by two real numbers $\langle m, a \rangle$. A car $\langle m, a \rangle$ is at least as good as a car $\langle m', a' \rangle$ if $m \leq m'$ and $a \geq a'$; in this case we write that $\langle m, a \rangle \succeq \langle m', a' \rangle$. Prove that \succeq is a partial order.

simple (just go over the three properties).

9. Determine the number of ordered pairs (A, B) , where $A \subseteq B \subseteq \{1, 2, \dots, n\}$.

Sketch. For each A compute the number of subsets of $\{1, \dots, n\}$ that contain A . Clearly there are 2^n subsets in total. Let $|A| = a$. The number of subsets of $\{1, \dots, n\}$ that include A should be 2^{n-a} .

It follows that the number we are looking for must be the sum of $2^{n-|A|}$ for all A , where A is a subset of $\{1, \dots, n\}$; this is written as $\sum_{k=0}^n \binom{n}{k} 2^{n-k}$. Using the binomial theorem we obtain that $2^n(1 + 1/2)^n = 3^n$. \square

10. We have k balls, and we distribute them into n (numbered) bins. Fill out the formulas for the number of distributions for various variants of the problem in the following table:

| | At most 1 ball into each bin | Any number of balls into each bin |
|---|------------------------------|-----------------------------------|
| Balls are distinguishable (have distinct colors) | | |
| Balls are indistinguishable | | |

Proof. We have n kinds of objects: x_1, \dots, x_n corresponding to the bins, the balls are kind of the positions we place the objects on. The cases on the tables are corresponding to permutation or combination where repetition is also considered.

| | At most 1 ball into each bin | Any number of balls into each bin |
|---|------------------------------|-----------------------------------|
| Balls are distinguishable (have distinct colors) | $n(n-1) \dots (n-k+1)$ | n^k |
| Balls are indistinguishable | $\binom{n}{k}$ | (multiset) $\binom{k+n-1}{n-1}$ |

for “multisets” consider the fact that a solution is of the form $\{x_1x_1x_3x_3x_4x_4x_4\}$ when we have $n = 4$ and $k = 6$. It is easy to see that this corresponds a possible solution of the equation $i_1 + \dots + i_n = k$. (cf. page 373-374)