CSE 237, Spring 2006
SOLUTION: Homework 5, Problem 3.

To prove: the class of RLs is a *strict* subset of the class of CFLs.

There are two parts to this proof, (A) that every RL is a CFL, and (B) that some CFLs are not RLs.

A. To prove: every regular language is context-free.

1) Let L_0 be a regular language.
2) By definition of regular language, we know there exists a regular expression E_0 such that $L(E_0) = L_0$.
3) **Our claim: there exists a context-free grammar G_0 such that $L(G_0) = L(E_0)$.** Our proof of this claim will proceed by structural induction on E_0.

Base case 1: $E_0 = \boldsymbol{\varepsilon}$. In this case, $L(E_0) = \{ \varepsilon \}$.

We have shown in class that exists a context-free grammar G_0 such that $L(G_0) = \{ \varepsilon \}$. QED.

Base case 2: $E_0 = \varnothing$. In this case, $L(E_0) = \varnothing$.

We have shown in class that exists a context-free grammar G_0 such that $L(G_0) = \varnothing$. QED.

Base case 3: $E_0 = \mathbf{a}$, where **a** is an arbitrary member of Σ. In this case, $L(E_0) = \{ \text{"a"} \}$.

We have shown in class that exists a context-free grammar G_0 such that $L(G_0) = \{ \text{"a"} \}$. QED.

This takes care of the three base cases. Now there will be three more cases for E_0, for which we need to make the following inductive assumption:

Inductive Assumption: Our claim holds for any subexpression E' of E_0. That is, if E' is some subexpression of E_0, then exists a context-free grammar that is equivalent to E'.

Case 1: $E_0 = E_1 \cup E_2$. In this case, $L(E_0) = L(E_1) \cup L(E_2)$.

Let G_1 and G_2 be context-free grammars such that $L(G_1) = L(E_1)$ and $L(G_2) = L(E_2)$. (By inductive assumption, they must exist). We've shown in class that there exists a context-free grammar G_0 such that $L(G_0) = L(G_1) \cup L(G_2)$. Therefore, $L(G_0) = L(E_1) \cup L(E_2) = L(E_0)$. QED.

Case 2: $E_0 = E_1 E_2$. In this case, $L(E_0) = L(E_1) \circ L(E_2)$.

Let G_1 and G_2 be context-free grammars such that $L(G_1) = L(E_1)$ and $L(G_2) = L(E_2)$. (By inductive assumption, they must exist). We've shown in class that there exists a context-free grammar G_0 such that $L(G_0) = L(G_1) \circ L(G_2)$. Therefore, $L(G_0) = L(E_1) \circ L(E_2) = L(E_0)$. QED.

Case 3: $E_0 = E_1^*$. In this case, $L(E_0) = L(E_1)^*$.

Let G_1 be a context-free grammar such that $L(G_1) = L(E_1)$. (By inductive assumption, it must exist). We've shown in class that there exists a context-free grammar G_0 such that $L(G_0) = L(G_1)^*$. Therefore, $L(G_0) = L(E_1)^* = L(E_0)$. QED.

4) We have thus proved our claim. Therefore, for any regular language L_0 there exists a context-free grammar G_0 such that $L(G_0) = L_0$.
5) This implies that every regular language is context free.

B. To prove: some context-free languages are not regular.

1) Consider the language $\{a^n b^n : n >= 0\}$.
2) We have shown in class that this language is not regular, by using the pumping lemma.
3) We have also shown that it is context free, by constructing a context-free grammar for it. QED.