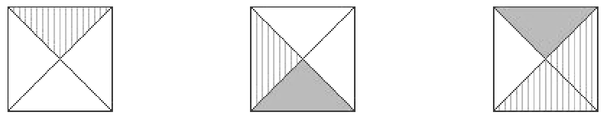
Example 1: tiling problems

Consider the set of three tiles below:

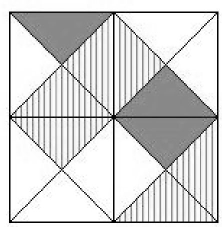


(1) (2) (3)

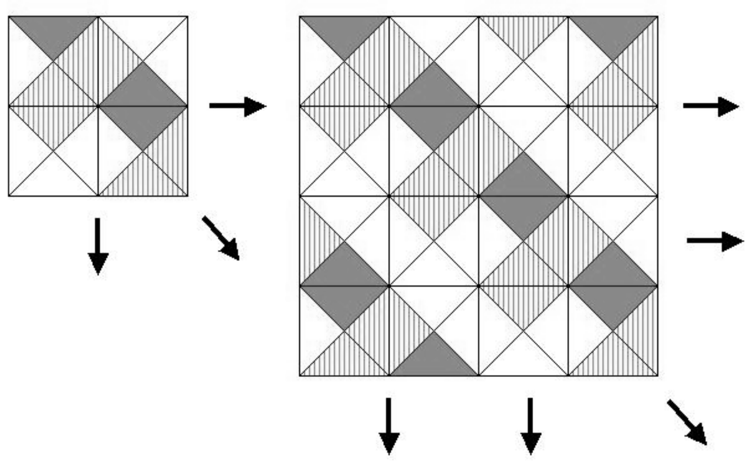
Could these be used to tile an arbitrary n×n area? The rules that have to be obeyed are:

- Only these three tile types can be used, but each can be used arbitrarily often.
- The edges of the tiles have to match up when they are used to cover an area.
- The tiles can’t be rotated.

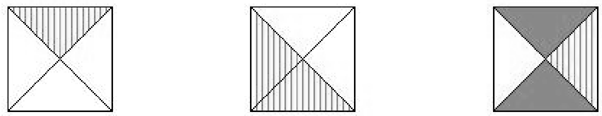
It’s clear that they can when n=2:



Also, in this simple case, one can see that the solution above extends easily to any n×n area.

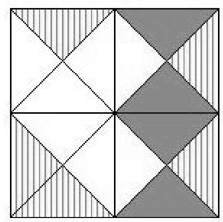


Now suppose that the bottom colours of tiles (2) and (3) are exchanged:

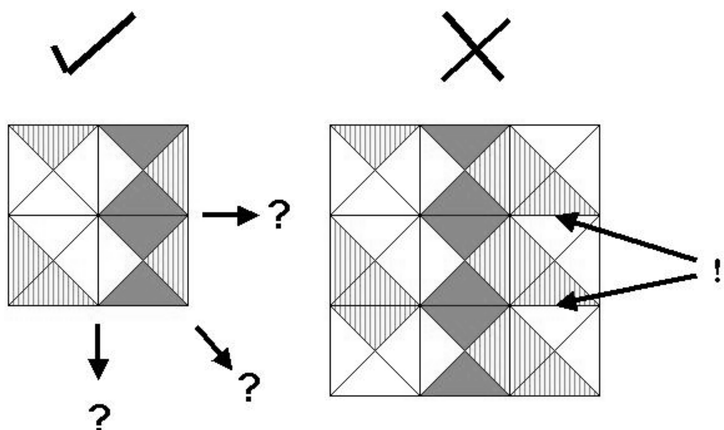


(1) (2) (3)

The new set of tiles can still cover a 2×2 area as below



but this particular solution can't be extended:



Moreover, there are no others which will work; this set can't tile any area of size greater than 2×2.

The most common forms of the 'tiling problem' are a generalisation of the problem above, with a supplied set T of tile types (which can't be added to or modified).

Bounded tiling problem:

Given a set T of tiles, can these be used to cover a specified n×n area?

This problem is in fact in the set NPC: it has a short certificate (a putative tiling of the n×n area can be checked (to see that the 'matching edges' and 'no rotations' rules have been obeyed) in time in $O(n^2)$, so is in NP; the problem also can be shown to be such that $B \leq_p \text{(bounded tiling)}$ for some $B \in NPC$, so bounded tiling is NP-complete. There appears to be no algorithm that can do significantly better than just checking through all possible arrangements of the tiles, something that takes an exponential amount of time.

Unbounded tiling problem:

Given a set T of tiles, can these be used to cover <u>any</u> n×n area? (Or equivalently, can they be used to tile the <u>entire integer grid</u>?)

This problem is much worse than bounded tiling; it is in fact undecidable, as the display of no finite tiled area can prove – in general, we aren't just talking about especially easy tile sets such as the one first considered above – that *any* area can be tiled.

However it would be wrong to assume that the essence of the difficulty was that the number of things that might need to be checked (all possible n×n areas) was infinite. It might have been that there was some rule that could look at a set of tiles and say "Yes, these have Property X, and so they can tile any area".